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Positivity of certain sums over Jacobi kernel polynomials

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Abstract

We present a computer-assisted proof of positivity of sums over kernel polynomials for ultraspherical Jacobi polynomials.

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1. Introduction

In this paper we show positivity of sums over Jacobi kernel polynomials $k_j^{\alpha}(x,0)$ on the interval $[-1,1]$ where we consider ultraspherical Jacobi polynomials $P_n^{(\alpha,\alpha)}(x)$ with $\alpha \in [-\frac{1}{2}, \frac{1}{2}]$. This problem originated in a new convergence proof for a certain finite element scheme in the course of which Schöberl [10] was led to conjecture the inequality

$$\sum_{j=0}^{n}(4j+1)(2n-2j+1)P_{2j}(0)P_{2j}(x) \geqslant 0 \tag{1}$$

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for $-1 \leqslant x \leqslant 1$ and $n \geqslant 0$, where $P_n(x)$ denotes the nth Legendre polynomial. This inequality corresponds to setting $\alpha = 0$ in the inequality of Theorem 1 that will be proven below. No human proof for this special case is known. Even asymptotics seem to be difficult [7].

In this paper we present a proof that makes heavy use of computer algebra. Based on treating the special cases $\alpha = \pm\frac{1}{2}$ we determine a decomposition of the given sum into expressions that can be estimated from below. For this proof we use the Mathematica packages `SumCracker` [8] and `GeneratingFunctions` [9]. Both implementations, as well as a variety of other algorithms for symbolic summation are available at

`http://www.risc.uni-linz.ac.at/research/combinat/software/`

In the following section we introduce kernel polynomials and formulate the conjectured inequality. We also outline the background from which the original problem (1) emerged. In Section 3 we show positivity for the special cases $\alpha = \pm\frac{1}{2}$ when $P_n^{(\alpha,\alpha)}(x)$ are Chebyshev polynomials. This proof motivates a decomposition of the given sum in the remaining case $-\frac{1}{2} < \alpha < \frac{1}{2}$, Lemma 4 in Section 4, which allows to find a lower bound in closed form whose positivity can be verified using `SumCracker`'s `ProveInequality` command.

2. Motivation

When constructing a smoothing operator for a high order finite element scheme, Schöberl [10] considered an integral operator that serves as point evaluation when applied to polynomials up to a given degree n. More precisely, he wanted to find a family of polynomials $\{\phi_n\}$ such that

$$\int_{-1}^{1} \phi_n(x) v(x)\, dx = v(0), \tag{2}$$

for all polynomials v with $\deg v \leqslant n$. Moreover, he wanted $\{\phi_n\}$ to satisfy the norm estimate

$$\|\phi_n\|_{L^1} = \int_{-1}^{1} \left|\phi_n(x)\right| dx \leqslant C,$$

where the constant C is independent of n. Property (2) led to consider so-called kernel polynomials.

Let $\{p_j(x)\}$ be a given sequence of polynomials defined on a real interval $[a, b]$ and being orthogonal with respect to some weight function $w(x) : [a, b] \to \mathbb{R}$. Then the kernel polynomial sequence is defined as

$$k_n(x, y) = \sum_{j=0}^{n} \frac{1}{h_j} p_j(x) p_j(y), \tag{3}$$

where $h_n = \int_a^b p_n(x)^2 w(x)\,dx$. Kernel polynomials have the reproducing property

$$\int_a^b k_n(x,y)q(x)w(x)\,dx = q(y)$$

for all polynomials $q(x)$ with degree less or equal to n. From the three-term recurrence relation for the $p_n(x)$ one easily obtains a compact expression for these kernel polynomials, namely

$$k_n(x,y) = c(n)\frac{p_{n+1}(x)p_n(y) - p_{n+1}(y)p_n(x)}{x-y},$$

where $c(n)$ depends on h_n and the leading coefficients of $p_{n+1}(x)$ and $p_n(x)$, for more details see e.g. [1,11].

In the following we consider only kernel polynomials for Jacobi polynomials of the form $P_n^{(\alpha,\alpha)}(x)$ which we denote by $k_n^\alpha(x,y)$. Jacobi polynomials $P_n^{(\alpha,\alpha)}(x)$ are orthogonal with respect to the weight function $w(x) = (1-x^2)^\alpha$. Their kernel polynomials can be expressed as

$$k_n^\alpha(x,y) = \frac{c_n^\alpha}{x-y}\big[P_{n+1}^{(\alpha,\alpha)}(x)P_n^{(\alpha,\alpha)}(y) - P_n^{(\alpha,\alpha)}(x)P_{n+1}^{(\alpha,\alpha)}(y)\big], \tag{4}$$

where

$$c_n^\alpha = 2^{-2\alpha-1}\frac{\Gamma(n+2)\Gamma(n+2\alpha+2)}{\Gamma(n+\alpha+1)\Gamma(n+\alpha+2)}.$$

If we choose ϕ_n to be the Legendre kernel polynomials $k_n^0(x,0)$ then condition (2) is satisfied because of the reproducing property with respect to the L^2-inner product $\int_{-1}^1 f(x)g(x)\,dx$ corresponding to the constant weight function $w(x) = (1-x)^0 \equiv 1$. But numerical computations suggest that the $k_n^0(x,0)$ are not uniformly bounded in the L^1-norm. So Schöberl was led to consider a modified ansatz using so-called gliding averages [4],

$$\phi_n(x) = \frac{1}{n+1}\sum_{j=n}^{2n} k_j^0(x,0). \tag{5}$$

Here ϕ_n is a polynomial of degree $2n$ satisfying (2). Defining the sum

$$S(n,x) = \frac{1}{n+1}\sum_{j=0}^{n} k_j^0(x,0), \tag{6}$$

we can write ϕ_n in the form

$$\phi_n(x) = \frac{2n+1}{n+1}S(2n,x) - \frac{n}{n+1}S(n-1,x).$$

Schöberl conjectured that (6) is positive for even indices, i.e. $S(2n,x) \geqslant 0$. If this is true, then one can bound the L^1-norm of ϕ_n for odd n immediately via

$$\begin{aligned}\|\phi_n\|_{L^1} &\leqslant \frac{2n+1}{n+1}\int_{-1}^{1} S(2n,x)\,dx + \frac{n}{n+1}\int_{-1}^{1} S(n-1,x)\,dx \\ &= \frac{3n+1}{n+1} \leqslant 3, \quad n \text{ odd.}\end{aligned}$$

Here we only needed to invoke the positivity of $S(2n,x)$ and its constant preserving property. After applying the triangle inequality we can omit the absolute values and evaluate each of the integrals over $S(2n,x)$ and $S(n-1,x)$ to 1. Having only an estimate for ϕ_{2n+1} at hand clearly is no obstruction to the application we have in mind since the degree of the smoothing operator can always be raised by one, if needed.

Trying to prove that $S(2n,x) \geqslant 0$, $x \in [-1,1]$, we observed that this inequality seems to remain valid if we consider more general sums over Jacobi kernel polynomials k_n^α with $\alpha \in [-\frac{1}{2}, \frac{1}{2}]$. Consequently we define

$$S_n^\alpha(x,y) := \sum_{j=0}^{n} k_j^\alpha(x,y).$$

In this notation we have $S(n,x) = (n+1)\, S_n^0(x,0)$. In the remainder of this paper we will prove the extended conjecture formulated in the following theorem.

Theorem 1. *For $-\frac{1}{2} \leqslant \alpha \leqslant \frac{1}{2}$, $-1 \leqslant x \leqslant 1$, $n \geqslant 0$, we have $S_{2n}^\alpha(x,0) \geqslant 0$.*

Note that for odd degrees, i.e. $S_{2n+1}^\alpha(x,0)$, the sums are not positive. Using the definition (3) of kernel polynomials, $S_n^\alpha(x,y)$ can be written as the single sum

$$S_n^\alpha(x,y) = \sum_{i=0}^{n} \frac{n-i+1}{h_i^\alpha} P_i^{(\alpha,\alpha)}(x) P_i^{(\alpha,\alpha)}(y).$$

The positivity of trigonometric series as well as their generalizations to Jacobi polynomial series has been considered in many other areas of mathematics. One famous example for an inequality of this kind is the Askey–Gasper inequality for the sum $\sum_{k=0}^n P_k^{(\alpha,\beta)}(x)/P_k^{(\beta,\alpha)}(1)$, see [1,2,5]. For $\beta = 0$ this sum can be expressed as the square of a hypergeometric function using a formula of Clausen. For $\beta \geqslant 0$ and $\alpha + \beta > -1$ positivity follows from this result by using a integral representation of Jacobi polynomials. This case also includes Fejér's inequality $\sum_{k=0}^n P_k(x) \geqslant 0$.

Another related problem discussed in [2] is determining when the sums

$$\sum_{k=0}^{n} \frac{(\gamma+1)_{n-k}}{(n-k)!} \frac{(2k+\alpha+\beta+1)(\alpha+\beta+1)_k}{k!} \frac{P_k^{(\alpha,\beta)}(x)}{P_k^{(\alpha,\beta)}(1)}$$

are non-negative for $-1 \leqslant x \leqslant 1$. In the ultraspherical case $\alpha = \beta$ with $\gamma = 2\alpha + 3$ non-negativity can be proven by showing that the generating functions of these sums are products of absolutely monotonic functions, cf. [1] and references therein. However, none of the techniques mentioned so far are applicable to proving Theorem 1, at least not directly.

The proof of Theorem 1 will be split into two parts. In Section 3 we will consider the cases $\alpha = \pm\frac{1}{2}$, corresponding to the Chebyshev polynomials of the first and second kind, respectively. The proof of these special cases motivates a decomposition of the sum $S_{2n}^{\alpha}(x,0)$ which is the key to proving Theorem 1 for the remaining part where $-\frac{1}{2} < \alpha < \frac{1}{2}$.

3. Chebyshev polynomials of first and second kind ($\alpha = \pm\frac{1}{2}$)

Scaled Jacobi polynomials $\sqrt{\pi} n!/\Gamma(n+\frac{1}{2}) P_n^{(-1/2,-1/2)}(x)$ are identical to Chebyshev polynomials of the first kind $T_n(x)$. The sum $S_n^{-1/2}(x,y)$ is called Fejér kernel and positivity is well known for all $n \geqslant 0$ and for all x, y in the unit square $[-1,1]^2$, for a short proof see e.g. [12]. Hence we only have to consider the case $\alpha = \frac{1}{2}$.

For $\alpha = \frac{1}{2}$ Jacobi polynomials $\sqrt{\pi}/2\,\Gamma(n+2)/\Gamma(n+\frac{3}{2}) P_n^{(1/2,1/2)}(x)$ are called Chebyshev polynomials of the second kind and commonly denoted by $U_n(x)$. Their kernel polynomials are

$$k_n^{1/2}(x,y) = \frac{1}{\pi(x-y)}\bigl[U_{n+1}(x)U_n(y) - U_n(x)U_{n+1}(y)\bigr].$$

`SumCracker` yields a closed form for $S_n^{1/2}(x,y)$, namely,

$$S_n^{1/2}(x,y) = \frac{1}{\pi(x-y)^2}\bigl[U_{n+1}(x)\bigl(xU_n(y) - U_{n+1}(y)\bigr) + U_n(x)\bigl(yU_{n+1}(y) - U_n(y)\bigr) + 1\bigr]. \tag{7}$$

Remark 2. Here we used the `Crack` command which takes an expression and returns a reformulation in "smaller" terms. A "human" proof of this identity which only uses the Chebyshev three-term recurrence will be given later in this section.

To prove that $S_{2n}^{1/2}(x,0) \geqslant 0$ we proceed as follows. Since $U_{2n+1}(0) = 0$ and $U_{2n}(0) = (-1)^n$ we have that

$$S_{2n}^{1/2}(x,0) = \frac{1}{\pi x^2}\bigl[1 + (-1)^n x\, U_{2n+1}(x) - (-1)^n U_{2n}(x)\bigr].$$

Inspection of the first few polynomials $S_{2n}^{1/2}(x,0)$ suggests that

$$S_{4m}^{1/2}(x,0) = p_{2m}(x)^2 \quad \text{and} \quad S_{4m+2}^{1/2}(x,0) = \bigl(1-x^2\bigr)q_{2m}(x)^2,$$

where $p_{2m}(x)$, $q_{2m}(x)$ are polynomials of degree $2m$ satisfying the relation $q_n(x)S_1^{1/2}(x,0) = (p_{n+1}(x) - p_n(x))^2$. To verify this claim we first use the `GuessRE` command of Mallinger's `GeneratingFunctions` package that tries to guess a holonomic recurrence equation given the first few terms of a sequence. Applying this function to $p_n(x)$ yields a recurrence relation that can easily be identified as the three-term recurrence for Chebyshev polynomials of the first kind. This rewriting of $S_{2n}^{1/2}(x,0)$ found by guessing can then easily be proven either by hand or invoking again computer algebra.

Lemma 3. *For $m \geqslant 0$ and $-1 \leqslant x \leqslant 1$ we have*

$$S_{4m}^{1/2}(x,0) = \frac{2}{\pi x^2} T_{2m+1}(x)^2,$$

and

$$S_{4m+2}^{1/2}(x,0) = \frac{1}{2\pi x^2(1-x^2)}\bigl(T_{2m+3}(x) - T_{2m+1}(x)\bigr)^2,$$

where $T_m(x)$ are the Chebyshev polynomials of the first kind.

Proof. The closed forms for $S_{4m}^{1/2}(x,0)$ and $S_{4m+2}^{1/2}(x,0)$ can be verified immediately with Kauers' `SumCracker` package. For this purpose we use an algorithm that decides zero equivalences of a given admissible sequence, for details see [8]. To prove the identities use the `ZeroSequenceQ` command with input

$$\begin{aligned}&\texttt{ZeroSequenceQ}\bigl[x\,\texttt{ChebyshevU}[4m+1,x] - \texttt{ChebyshevU}[4m,x] + 1\\ &\quad - 2\,\texttt{ChebyshevT}[2m+1,x]^2\bigr]\end{aligned}$$

and

$$\begin{aligned}&\texttt{ZeroSequenceQ}\bigl[-x\,\texttt{ChebyshevU}[4m+3,x] + \texttt{ChebyshevU}[4m+2,x]\\ &\quad + 1 - \frac{1}{2(1-x^2)}\bigl(\texttt{ChebyshevT}[2m+3,x] - \texttt{ChebyshevT}[2m+1,x]\bigr)^2\bigr]\end{aligned}$$

This immediately yields `True` in both cases. □

From these representations it is obvious that the sums $S_{2n}^{1/2}(x,0)$ are non-negative. While there exists a closed form representation of $S_n^{1/2}(x,y)$, there is no closed form of $S_n^{\alpha}(x,y)$ for general α. Still, examining a derivation of (7) using only the three-term recurrence satisfied by $U_n(x)$ indicates how to continue dealing with general Jacobi polynomials $P_n^{(\alpha,\alpha)}(x)$, $-\frac{1}{2} < \alpha < \frac{1}{2}$.

So, let again $\alpha = \frac{1}{2}$. In order to derive (7), we show that $S_n^{1/2}(x,y)$ rewritten according to (4) as the sum

$$S_n^{1/2}(x,y) = \frac{1}{\pi(x-y)} \sum_{j=0}^{n} \bigl[U_{j+1}(x)U_j(y) - U_j(x)U_{j+1}(y)\bigr],$$

is a sum representation which telescopes to the right-hand side of (7). Because of symmetry it suffices to consider only one part of the sum. For the first part, `SumCracker` yields

$$\begin{aligned}(x-y)\sum_{j=0}^{n} U_{j+1}(x)U_j(y) &= \frac{1}{2}\bigl(2xU_{n+1}(x)U_n(y) - U_n(x)U_n(y)\\ &\quad - U_{n+1}(x)U_{n+1}(y) + 1\bigr),\end{aligned}$$

which implies

$$(x-y)U_{j+1}(x)U_j(y)=\frac{1}{2}\Delta_j\big(\underbrace{2xU_j(x)U_{j-1}(y)-U_{j-1}(x)U_{j-1}(y)-U_j(x)U_j(y)}_{:=G_j(x,y)}\big),$$

where Δ_j denotes the difference operator $\Delta_j[\psi(j)]=\psi(j+1)-\psi(j)$. The correctness of this identity can be verified by straightforward calculation using the three-term recurrence for Chebyshev polynomials,

$$U_n(x)-2xU_{n+1}(x)+U_{n+2}(x)=0, \qquad U_0(x)=1, \qquad U_1(x)=2x. \tag{8}$$

Namely, first we use (8) to rewrite $2xU_j(x)$ and then, to involve y, we use the same recurrence relation to replace $U_{j-1}(y)+U_{j+1}(y)$. This way we obtain

$$\begin{aligned}G_{j+1}(x,y)-G_j(x,y)&=2xU_j(y)U_{j+1}(x)-U_{j+1}(x)U_{j+1}(y)\\&\quad-2xU_{j-1}(y)U_j(x)+U_{j-1}(x)U_{j-1}(y)\\&=2xU_j(y)U_{j+1}(x)-U_{j+1}(x)U_{j+1}(y)\\&\quad-U_{j-1}(y)U_{j+1}(x)\\&=2(x-y)U_{j+1}(x)U_j(y).\end{aligned} \tag{9}$$

Note that this telescoper has to exist because Chebyshev polynomials satisfy a three-term recurrence with *constant* coefficients. The procedure above cannot be generalized to Jacobi polynomials $P_n^{(\alpha,\alpha)}(x)$, $\alpha\neq\pm\frac{1}{2}$, because the *polynomial* recurrence coefficients do not enable appropriate cancellation in this case. However mimicking the steps of the proof above one obtains a decomposition of $S_{2n}^\alpha(x,0)$, $-\frac{1}{2}<\alpha<\frac{1}{2}$, that makes the problem better treatable with our methods.

We remark that because of the fact that Chebyshev polynomials of first and second kind satisfy the same recurrence relation but with different starting values, a closed form for $S_n^{-1/2}(x,y)$ can be computed completely analogously.

4. Jacobi polynomials $P_n^{(\alpha,\alpha)}(x)$ with $-\frac{1}{2}<\alpha<\frac{1}{2}$

In this section we prove Theorem 1, i.e. the positivity of $S_{2n}^\alpha(x,0)$, $-\frac{1}{2}<\alpha<\frac{1}{2}$, where the sum representation according to (4) is given by

$$S_n^\alpha(x,y)=\frac{1}{x-y}\sum_{j=0}^n c_j^\alpha\big[P_{j+1}^{(\alpha,\alpha)}(x)P_j^{(\alpha,\alpha)}(y)-P_j^{(\alpha,\alpha)}(x)P_{j+1}^{(\alpha,\alpha)}(y)\big], \tag{10}$$

with

$$c_j^\alpha=2^{-2\alpha-1}\frac{\Gamma(j+2)\Gamma(j+2\alpha+2)}{\Gamma(j+\alpha+1)\Gamma(j+\alpha+2)}.$$

To this end we need several intermediate results starting with a suitable decomposition of $S_n^\alpha(x,y)$ which will be obtained by following the steps of the derivation (9). For this we will invoke the three-term recurrence [1,11]:

$$\begin{aligned}(n+2)(n+2\alpha+2)\,P_{n+2}^{(\alpha,\alpha)}(x) &= (n+\alpha+2)(2n+2\alpha+3)\,x\,P_{n+1}^{(\alpha,\alpha)}(x)\\ &\quad-(n+\alpha+1)(n+\alpha+2)\,P_n^{(\alpha,\alpha)}(x)\end{aligned} \tag{11}$$

for $n \geqslant 0$ and the initial values $P_{-1}^{(\alpha,\alpha)}(x)=0,\ P_0^{(\alpha,\alpha)}(x)=1$. With this relation we obtain for all $j \geqslant 0$,

$$\begin{aligned}
&(x-y)c_j^\alpha P_{j+1}^{(\alpha,\alpha)}(x)P_j^{(\alpha,\alpha)}(y)\\
&= xc_j^\alpha P_{j+1}^{(\alpha,\alpha)}(x)P_j^{(\alpha,\alpha)}(y) - \frac{c_j^\alpha}{(j+\alpha+1)(2j+2\alpha+1)}P_{j+1}^{(\alpha,\alpha)}(x)\\
&\quad\times\bigl[(j+\alpha)(j+\alpha+1)P_{j-1}^{(\alpha,\alpha)}(y)+(j+1)(j+2\alpha+1)P_{j+1}^{(\alpha,\alpha)}(y)\bigr]\\
&= xc_j^\alpha P_{j+1}^{(\alpha,\alpha)}(x)P_j^{(\alpha,\alpha)}(y) - c_j^\alpha\frac{(j+1)(j+2\alpha+1)}{(j+\alpha+1)(2j+2\alpha+1)}P_{j+1}^{(\alpha,\alpha)}(x)P_{j+1}^{(\alpha,\alpha)}(y)\\
&\quad- c_j^\alpha\frac{(j+\alpha)(j+\alpha+1)}{(2j+2\alpha+1)(j+1)(j+2\alpha+1)}P_{j-1}^{(\alpha,\alpha)}(y)\\
&\quad\times\bigl[x(2j+2\alpha+1)P_j^{(\alpha,\alpha)}(x)-(j+\alpha)P_{j-1}^{(\alpha,\alpha)}(x)\bigr]\\
&= xc_j^\alpha P_{j+1}^{(\alpha,\alpha)}(x)P_j^{(\alpha,\alpha)}(y) - xc_{j-1}^\alpha P_j^{(\alpha,\alpha)}(x)P_{j-1}^{(\alpha,\alpha)}(y)\\
&\quad- c_j^\alpha\frac{(j+1)(j+2\alpha+1)}{(j+\alpha+1)(2j+2\alpha+1)}P_{j+1}^{(\alpha,\alpha)}(x)P_{j+1}^{(\alpha,\alpha)}(y)\\
&\quad+ c_j^\alpha\frac{(j+\alpha)^2(j+\alpha+1)}{(j+1)(j+2\alpha+1)(2j+2\alpha+1)}P_{j-1}^{(\alpha,\alpha)}(x)P_{j-1}^{(\alpha,\alpha)}(y).
\end{aligned}$$

Now we plug this identity into definition (10), set $y=0$ and substitute $n\mapsto 2n$. This gives

$$\begin{aligned}
x^2S_{2n}^\alpha(x,0) &= \sum_{j=0}^{2n} x\,\Delta_j\bigl[c_{j-1}^\alpha P_j^{(\alpha,\alpha)}(x)P_{j-1}^{(\alpha,\alpha)}(0)\bigr]\\
&\quad- 2\sum_{j=0}^{2n} c_j^\alpha\frac{(j+1)(j+2\alpha+1)}{(j+\alpha+1)(2j+2\alpha+1)}P_{j+1}^{(\alpha,\alpha)}(x)P_{j+1}^{(\alpha,\alpha)}(0)\\
&\quad+ 2\sum_{j=0}^{2n} c_j^\alpha\frac{(j+\alpha)^2(j+\alpha+1)}{(j+1)(j+2\alpha+1)(2j+2\alpha+1)}P_{j-1}^{(\alpha,\alpha)}(x)P_{j-1}^{(\alpha,\alpha)}(0).
\end{aligned}$$

The first sum can easily be simplified by telescoping, the second and third sum can be combined by shifting summation indices. We also use the fact that ultraspherical Jacobi polynomials $P_n^{(\alpha,\alpha)}$ of odd degree vanish at $x=0$. Thus with

$$g_{2n}^\alpha(x,0) = c_{2n}^\alpha\left[x\,P_{2n+1}^{(\alpha,\alpha)}(x) - 2\frac{2n+\alpha+1}{4n+2\alpha+3}P_{2n}^{(\alpha,\alpha)}(x)\right]P_{2n}^{(\alpha,\alpha)}(0)$$

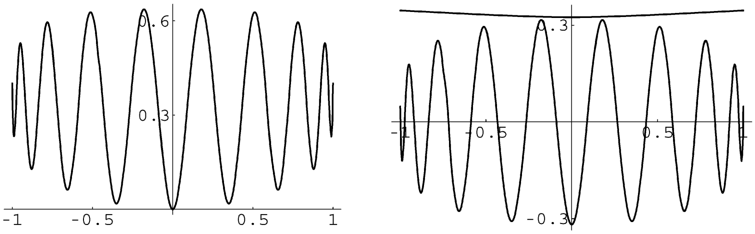

Fig. 1. $x^2 S^0_{2n}(x,0)$ and $f^0_{2n}(x,0)$, $g^0_{2n}(x,0)$ for $n = 8$.

and

$$f^\alpha_{2n}(x,0) = 2\bigl(4\alpha^2 - 1\bigr) \sum_{j=0}^{n} \frac{(2j+\alpha+1)c^\alpha_{2j}\, P^{(\alpha,\alpha)}_{2j}(0) P^{(\alpha,\alpha)}_{2j}(x)}{(2j+1)(2j+2\alpha+1)(4j+2\alpha-1)(4j+2\alpha+3)}$$

we obtain a decomposition of the sum $S^\alpha_{2n}(x,0)$. Note that for Chebyshev polynomials, i.e. $\alpha = \pm\frac{1}{2}$, $f^\alpha_{2n}(x,0)$ collapses to 0 because of the factor $(4\alpha^2 - 1)$. Only the closed form $g^\alpha_{2n}(x,0)$ survives.

Lemma 4.

$$x^2 S^\alpha_{2n}(x,0) = f^\alpha_{2n}(x,0) + g^\alpha_{2n}(x,0), \quad -\tfrac{1}{2} < \alpha < \tfrac{1}{2},\ -1 \leqslant x \leqslant 1,\ n \geqslant 0.$$

As can be seen from Fig. 1, $g^\alpha_{2n}(x,0)$ contains the main oscillations whereas in $f^\alpha_{2n}(x)$ they are dampened out. In order to prove non-negativity of $S^\alpha_{2n}(x,0)$ we will show that $f^\alpha_{2n}(x,0) + g^\alpha_{2n}(x,0) \geqslant 0$. This will be achieved by estimating the sum $f^\alpha_{2n}(x,0)$ from below. The sum of this lower bound and $g^\alpha_{2n}(x,0)$ can then be shown to be positive with `SumCracker`'s `ProveInequality` command.

The first step is to define, more generally, f^α_n for arguments $x, y \in [-1,1]$ by

$$f^\alpha_n(x,y) = 2\bigl(4\alpha^2 - 1\bigr) \sum_{j=0}^{n} \frac{(j+\alpha+1)c^\alpha_j P^{(\alpha,\alpha)}_j(x) P^{(\alpha,\alpha)}_j(y)}{(j+1)(j+2\alpha+1)(2j+2\alpha-1)(2j+2\alpha+3)}.$$

This definition is consistent with that of $f^\alpha_{2n}(x,0)$ above. The coefficient of the Jacobi polynomials inside the sum is positive for $j \geqslant 1$, hence we have

$$\sum_{j=1}^{n} \frac{(j+\alpha+1)c^\alpha_j}{(j+1)(j+2\alpha+1)(2j+2\alpha-1)(2j+2\alpha+3)} \bigl[P^{(\alpha,\alpha)}_j(x) - P^{(\alpha,\alpha)}_j(y)\bigr]^2 \geqslant 0,$$

which is equivalent to

$$-\sum_{j=0}^{n}\frac{2(j+\alpha+1)c_j^\alpha}{(j+1)(j+2\alpha+1)(2j+2\alpha-1)(2j+2\alpha+3)}P_j^{(\alpha,\alpha)}(x)P_j^{(\alpha,\alpha)}(y)$$
$$\geqslant -\sum_{j=0}^{n}\frac{(j+\alpha+1)c_j^\alpha}{(j+1)(j+2\alpha+1)(2j+2\alpha-1)(2j+2\alpha+3)}P_j^{(\alpha,\alpha)}(x)^2$$
$$-\sum_{j=0}^{n}\frac{(j+\alpha+1)c_j^\alpha}{(j+1)(j+2\alpha+1)(2j+2\alpha-1)(2j+2\alpha+3)}P_j^{(\alpha,\alpha)}(y)^2.$$

Since $(1-2\alpha)(1+2\alpha)$ is positive for $-\frac{1}{2}<\alpha<\frac{1}{2}$, both sides of the last inequality can be multiplied with this factor to obtain the following.

Lemma 5. *Let $-\frac{1}{2}<\alpha<\frac{1}{2}$. Then*

$$f_n^\alpha(x,y)\geqslant\tfrac{1}{2}\big(f_n^\alpha(x,x)+f_n^\alpha(y,y)\big),\quad n\geqslant 0,$$

for all $x,y\in[-1,1]$.

This lower bound has the advantage that we can find a closed form for $f_n^\alpha(x,x)$. Although Kauers' package `SumCracker` does not find a closed form of $f_n^\alpha(x,x)$ for symbolic α, for specific values of α it succeeds. Guessing on the coefficients of these expressions suggests the closed form stated in the next lemma. The key point, however, is discovering this identity. Once it has been found its validity can be proven fairly easily.

Lemma 6.

$$f_n^\alpha(x,x)=2c_n^\alpha\Bigg[\frac{(n+1)(n+2\alpha+1)}{(n+\alpha+1)(2n+2\alpha+1)}P_{n+1}^{(\alpha,\alpha)}(x)^2 - xP_n^{(\alpha,\alpha)}(x)P_{n+1}^{(\alpha,\alpha)}(x)+\frac{n+\alpha+1}{2n+2\alpha+3}P_n^{(\alpha,\alpha)}(x)^2\Bigg],$$

for all $n\geqslant 0$, $-1\leqslant x\leqslant 1$ and $\alpha>-1$. For $n=-1$ we have $f_{-1}^\alpha(x,x)=0$.

Proof. This identity can also be proven using `ZeroSequenceQ`. The coefficients c_n^α are given by their recurrence relation `cdef`, i.e.

$$\texttt{cdef}=\left\{c[k]==\frac{(k+1)(k+2\alpha+1)}{(k+\alpha)(k+\alpha+1)}c[k-1],\,c[0]==\frac{2^{-2\alpha-1}\Gamma[2\alpha+2]}{\Gamma[\alpha+1]\Gamma[\alpha+2]}\right\}.$$

The input form for `SumCracker` is

$$\texttt{ZeroSequenceQ}\Bigg[\big(4\alpha^2-1\big)$$
$$\texttt{SUM}\Bigg[\frac{(j+\alpha+1)c[j]}{(j+1)(j+2\alpha+1)(2j+2\alpha-1)(2j+2\alpha+3)}\texttt{JacobiP}[j,\alpha,\alpha,x]^2,\{j,0,n\}\Bigg]$$

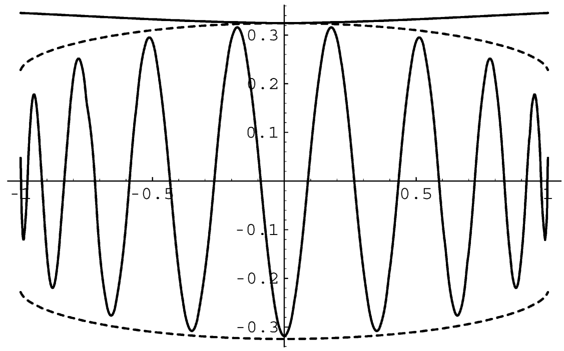

Fig. 2. Solid: $g_{2n}(x,0)$, $f_{2n}(x,0)$; dashed: $\pm\frac{1}{2}(f_{2n}(x,x)+f_{2n}(0,0))$.

$$-c[n]\biggl(\frac{(n+1)(n+2\alpha+1)}{(n+\alpha+1)(2n+2\alpha+1)}\texttt{JacobiP}[n+1,\alpha,\alpha,x]^2$$
$$-x\,\texttt{JacobiP}[n,\alpha,\alpha,x]\texttt{JacobiP}[n+1,\alpha,\alpha,x]$$
$$+\frac{n+\alpha+1}{2n+2\alpha+3}\texttt{JacobiP}[n,\alpha,\alpha,x]^2\biggr),\texttt{Where}\to\texttt{cdef}\biggr] \qquad \square$$

We remark that Lemma 6 can also be proven by showing that the closed form is the telescoper for the summand using only the Jacobi three-term recurrence. Fig. 2 illustrates how the functions $g^\alpha_{2n}(x,0)$, $f^\alpha_{2n}(x,0)$ and $\frac{1}{2}(f^\alpha_{2n}(x,x)+f^\alpha_{2n}(0,0))$ are related. Now we collect the previous lemmas to give a proof of Theorem 1.

Proof of Theorem 1. The cases $\alpha=\pm\frac{1}{2}$ are covered by the results of Section 3. For $\alpha=-\frac{1}{2}$ Theorem 1 follows from well-known results on the Fejér kernel [12] and positivity of $S^{1/2}_{2n}(x,0)$ is obvious from the rewriting stated in Lemma 3.

Next we consider $-\frac{1}{2}<\alpha<\frac{1}{2}$. With the decomposition given in Lemma 4 and the lower bound from Lemma 5 we have

$$x^2S^\alpha_{2n}(x,0)=g^\alpha_{2n}(x,0)+f^\alpha_{2n}(x,0)\geqslant g^\alpha_{2n}(x,0)+\frac{1}{2}\bigl(f^\alpha_{2n}(x,x)+f^\alpha_{2n}(0,0)\bigr).$$

To complete the proof it suffices to show positivity of the latter expression. (See Fig. 3.) Using Lemma 6 we have

$$\frac{1}{c^\alpha_{2n}}\biggl[g^\alpha_{2n}(x,0)+\frac{1}{2}\bigl(f^\alpha_{2n}(x,x)+f^\alpha_{2n}(0,0)\bigr)\biggr]$$
$$=\frac{(2n+1)(2n+2\alpha+1)}{(2n+\alpha+1)(4n+2\alpha+1)}P^{(\alpha,\alpha)}_{2n+1}(x)^2-xP^{(\alpha,\alpha)}_{2n+1}(x)\bigl[P^{(\alpha,\alpha)}_{2n}(x)-P^{(\alpha,\alpha)}_{2n}(0)\bigr]$$
$$+\frac{2n+\alpha+1}{4n+2\alpha+3}\bigl[P^{(\alpha,\alpha)}_{2n}(x)-P^{(\alpha,\alpha)}_{2n}(0)\bigr]^2. \tag{12}$$

We use the `ProveInequality` command of `SumCracker` in the following way:

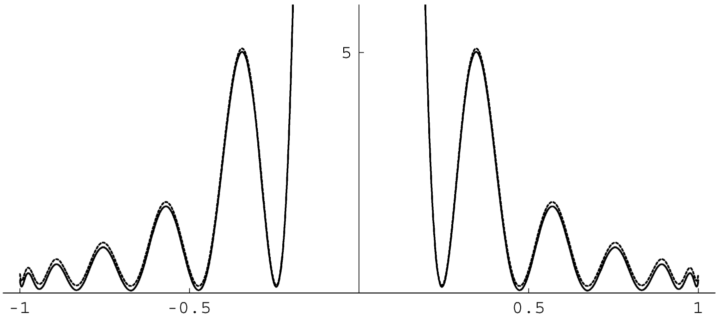

Fig. 3. $[g_{2n}(x,0) + \frac{1}{2}(f_{2n}(x,x) + f_{2n}(0,0))]/x^2$, dotted: $2S^0_{2n}(x,0)$ for $n = 12$.

$$\begin{aligned}
&\texttt{In[1]} := \texttt{ProveInequality}\Big[\frac{(2n+1)(2n+2\alpha+1)}{(2n+\alpha+1)(4n+2\alpha+1)}\texttt{JacobiP}[2n+1,\alpha,\alpha,x]^2 \\
&\quad - x\texttt{JacobiP}[2n+1,\alpha,\alpha,x]\big(\texttt{JacobiP}[2n,\alpha,\alpha,x] - \texttt{JacobiP}[2n,\alpha,\alpha,0]\big) \\
&\quad + \frac{2n+\alpha+1}{4n+2\alpha+3}\big(\texttt{JacobiP}[2n,\alpha,\alpha,x] - \texttt{JacobiP}[2n,\alpha,\alpha,0]\big)^2 \geqslant 0, \\
&\quad \texttt{Using} \to \big\{-1 \leqslant x \leqslant 1,\ -\tfrac{1}{2} < \alpha < \tfrac{1}{2}\big\}, \\
&\quad \texttt{Variable} \to n, \texttt{From} \to 0\Big] // \texttt{Timing} \\
&\texttt{Out[1]} = \{5358.25\texttt{Second}, \texttt{True}\} \qquad \square
\end{aligned}$$

The ProveInequality command constructs an inductive proof using cylindrical algebraic decomposition [3,6,8], which is also where the main computational effort lies.

5. Final remarks

The condition on α above cannot be removed if we want positivity of (12) for $n \geqslant 0$. It seems though that this expression stays non-negative for n greater than some lower bound, possibly depending on α.

An obvious open problem is to give a "human" proof of the positivity of the expression in (12).

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